Systems of parameters (In this section, all rings are Noetherian)

Using the PIT along with the corollary about primes minimal over an ideal in a local ring, we get the following characterization of dimension of a local ring:

Cor: If R is a local ring w/ maximal ideal $\mathfrak{m}$, then $\dim R$ is the smallest number d s.t. $\exists$ d elements $x_1, \dots, x_d \in \mathfrak{m}$ with $\mathfrak{m}^n \subseteq (x_1, \dots, x_d)$ for $n >> 0$.

Pf: If $\mathfrak{m}^n \subseteq (x_1, \dots, x_d) \subseteq \mathfrak{m}$ then $\mathfrak{m}$ is a minimal prime over $(x_1, \dots, x_d)$, so $\dim R \leq d$ by the PIT.

For the other inequality, we can find $x_1, \dots, x_e \in \mathfrak{m}$ with $e = \dim R$ s.t. $\mathfrak{m}$ is a minimal prime over $(x_1, \dots, x_e)$ by the converse of the PIT.

Then $R/(x_1, \dots, x_e)$ has only one prime ideal. Thus it is nilpotent by the Corollary in the previous section. Thus, $\dim R = e \geq d$, since d is the minimum # s.t. this works. $\square$

Def: If R is a local ring, then a sequence of elements $x_1, \dots, x_d$ as in the corollary (i.e. $d = \dim R$) is called a <u>system of parameters</u> for R.

Then if $(R, \mathfrak{m})$ is local of dimension d, we can deduce that the following are equivalent:

1.) $x_1, \ldots, x_d \in \mathfrak{m}$ is a system of parameters
2.) $\operatorname{rad}(x_1, \ldots, x_d) = \mathfrak{m}$
3.) $\mathfrak{m}$ is a minimal prime over $(x_1, \ldots, x_d)$

Recall that a local ring has finite length $\Leftrightarrow$ it's Artinian $\Leftrightarrow$ $\mathfrak{m}$ (the max'l ideal) is the only prime ideal $\Leftrightarrow$ $\mathfrak{m}^n = 0$ for $n \gg 0$.

Thus, if R is a local ring, $\mathfrak{m}^n \subset q$ for $n \gg 0$ $\Leftrightarrow$ R/q has finite length. Such an ideal q is said to have <u>finite colength</u>.

More generally, if M is a f.g. module over a local ring R, then an ideal $q \subseteq \mathfrak{m}$ has <u>finite colength on M</u> if M/qM has finite length.

Similar to the case of rings, this is true iff a power of $\mathfrak{m}$ annihilates M/qM.

i.e. $\mathfrak{m}^n \subseteq \operatorname{ann}(M/qM)$, which implies $\mathfrak{m} \subseteq \operatorname{rad}(\operatorname{ann}(\underbrace{M/qM}_{\neq 0 \text{ (Nakayama)}})) \neq R$

$\Rightarrow \mathfrak{m} = \operatorname{rad}(\operatorname{ann}(M/qM))$.

Claim: If R is any (not necessarily local) ring, M a finitely generated module, and $q \subseteq R$ an ideal, then $\text{rad}(\text{ann}(M/qM)) = \text{rad}(q + \text{ann}\, M)$.

Pf: Recall that the radical of an ideal is the intersection of all primes containing it, so it suffices to show that a prime P of R contains $\text{ann}(M/qM)$ iff P contains $q + \text{ann}\, M$.

Recall that $P \supseteq \text{ann}(M/qM) \iff (M/qM)_P = M_P/q_P M_P \neq 0$
(i.e. no unit annihilates it.)

By Nakayama, $M_P/q_P M_P \neq 0$ iff $M_P \neq 0$ and $q_P \subseteq P_P$.

These conditions are satisfied iff P contains both q and $\text{ann}\, M$, i.e. $P \supset q + \text{ann}\, M$. □

In the case of R local, this gives us the following more general version of the above corollary:

Prop: Let (R, m) be a local ring and M a finitely generated R-module. Let $q \subseteq R$ be an ideal. Then:

a.) q has finite colength on $M \iff (q + \text{ann}\, M) \supseteq m^n$ for $n >> 0 \iff q$ has finite colength on $R/\text{ann}M$

b.) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then q has finite colength on $M \iff q$ has finite colength on M' and M''.

c.) $\dim M$ is the least number d s.t. there is an ideal of finite colength on M generated by d elements.

Pf: a.) q has finite colength on M

$$\iff \operatorname{rad}(\operatorname{ann}(M/qM)) = m \iff m^n \subseteq q + \operatorname{ann} M, \quad n \gg 0.$$

$$\operatorname{rad}(\operatorname{ann}(M/qM)) = \operatorname{rad}(q + \operatorname{ann} M)$$

Note that $(R/\operatorname{ann} M) \big/ q\,(R/\operatorname{ann} M) = (R/\operatorname{ann} M) \Big/ \left((q + \operatorname{ann} M)/\operatorname{ann} M\right)$

$$\cong R/(q + \operatorname{ann} M), \text{ which}$$

has annihilator $q + \operatorname{ann} M$.

Thus q has finite colength on $R/\operatorname{ann} M \iff \operatorname{rad}(q + \operatorname{ann} M) = m$ and we've done by above.

b.) If q has finite colength on M, then

$$\operatorname{ann} M \subset \operatorname{ann} M' \cap \operatorname{ann} M'', \text{ so}$$

$q + \operatorname{ann} M \subseteq q + \operatorname{ann} M'$ and $q + \operatorname{ann} M \subseteq q + \operatorname{ann} M''$.

$\Rightarrow$ since $\text{rad}(\text{ann}\, M/qM)$ is maximal, the radicals of $\text{ann}\, M'/qM'$ and $\text{ann}\, M''/qM''$ must be as well.

For the converse, assume q has finite colength on M' and M''.

Tensoring the short exact sequence by R/q, we get

$$M'/qM' \to M/qM \to M''/qM'' \to 0$$

Thus the outer terms have finite length, so M/qM does as well.

c.) By definition, $\dim M = \dim R/\text{ann}\, M$

$\underset{\text{by cor}}{=}$ smallest # d s.t. $q = (x_1, \ldots, x_d)$ has finite colength on $R/\text{ann}M$ $n >> 0$.

By a.), we're done since q has finite colength on $M \Leftrightarrow$ it has finite colength on $R/\text{ann}\, M$. $\square$

In the local case, we get a sort of principal ideal theorem for dimension rather than codim:

Cor: If (R, m) is local and M a finitely generated R-module, then for any $x \in m$, we have

$$\dim M - 1 \leq \dim M/xM \leq \dim M.$$

Pf: The second inequality is clear since $\operatorname{ann} M \subseteq \operatorname{ann} M/xM$.

For the first, set $d = \dim M/xM$. Then $\exists\, q = (x_1, \ldots, x_d)$ of finite colength on M/xM.

Thus, $M/(x,q)M = M/(x_1, \ldots, x_d, x)M$ has finite length, so

$(x_1, \ldots, x_d, x)$ has finite colength on $M \Rightarrow \dim M \leq d+1$

$\Rightarrow \dim M - 1 \leq \dim M/xM$. $\square$